Grade 11/12 Math Circles

November 30, 2022

Generating Functions 2 - Problem Set

1. Create combinatorial classes and corresponding generating functions for the following situations:
   (a) 0-1 strings where we wish to count by the length of the strings.
   (b) 0-1-2 strings where we wish to count by the length of the strings.
   (c) Strings with k possible number entries, where we wish to count by the length of the strings.
   (d) Drawing socks out of a basket, where there are 3 red socks, 5 blue socks, 10 purple socks and 12 green socks.

2. Recall that $[z^n]F(z)$ represents the coefficient of z^n in the generating function $F(z)$. From your answers to Problem 1, find an expression for the following coefficients and describe each coefficient represents:
   (a) $[z^{12}]F(z)$, where $F(z)$ is the generating function from Problem 1a.
   (b) $[z^{40}]F(z)$, where $F(z)$ is the generating function from Problem 1b.
   (c) $[z^n]F(z)$, where $F(z)$ is the generating function from Problem 1c.

3. Find a generating function for 0-1-2 strings which start with 012.

4. Find an expression for the number of ways to make \$2.65 in change (with 5, 10 and 25 cent coins available, as well as 1 and 2 dollar coins).

5. Find two 4-sided dice such that:
   - Each side has a positive integer number of dots
   - The two dice are not the same
   - The probability of rolling a sum of $2, \ldots, 8$ on these dice is the same as the probabilities for regular 4-sided dice

   Hint: $(z + z^2 + z^3 + z^4)^2 = (z^2 + 1)^2(z + 1)^2 z^2$

6. **Challenge:** For any two n-sided fair dice ($n \geq 2$), will there always exist two other n-sided dice such that:
   - Each side has a positive integer number of dots
   - The two dice are not the same
   - The probability of rolling a sum of $2, 3, 4, \ldots, 2n$ on these dice is the same as the probabilities for two fair n-sided dice

7. How many compositions of n have k parts, where each part is an odd number?
8. Find the generating function for compositions of n which have k parts, where each part is at most 3.
9. Find the generating function for compositions of n which have 1 or 2 parts.
10. **Challenge:** How many compositions of n are there (of any number of parts)?